Math 113 (Calculus II)
Test 4 Form A KEY

Multiple Choice. Fill in the answer to each problem on your scantron. Make sure your name, section and instructor is on your scantron.

1. A power series for the function $\dfrac{3}{3-x}$ is given by:

   a) $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{3^n}$ b) $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^n}{3^n}$ c) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{x^n}{3^n}$

   d) $\displaystyle\sum_{n=0}^{\infty} 3^n x^n$ e) $\displaystyle 3\sum_{n=0}^{\infty} x^n$ f) none of the above

2. Find a power series representation for $f(x) = \ln\left(\dfrac{1+x}{1-x}\right)$.

   a) $2 + 2x + \dfrac{2x^3}{3} + \dfrac{2x^5}{5} + \cdots$ b) $2 + \dfrac{2x^2}{4} + \dfrac{2x^4}{4} + \cdots$ c) $2x + \dfrac{2x^3}{3} + \dfrac{2x^5}{5} + \cdots$

   d) $x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \cdots$ e) $x + \dfrac{x^2}{2} + \dfrac{x^3}{3} + \cdots$ f) none of the above.

3. Consider the parametric curve given by $x = e^t - 1$, $y = e^{2t}$. Eliminate the parameter to find a Cartesian equation of the curve.

   a) $y = x^2$ b) $y = x^2 + 2x$ c) $y = x^2 - 2x$

   d) $y = x^2 + 2x + 1$ e) $y = x^2 - 2x + 1$ f) None of the above.

4. Find a power series representation of the function

   $$f(x) = \frac{5}{16 + x^2}$$

   and determine its interval of convergence.

   a) $\displaystyle 5\sum_{n=0}^{\infty} (-1)^n x^{2n} \quad |x| < 1$

   b) $\displaystyle \frac{5}{16}\sum_{n=0}^{\infty} \frac{x^{2n}}{4^{2n}} \quad |x| < 4$

   c) $\displaystyle \frac{5}{16}\sum_{n=0}^{\infty} (-1)^n x^{2n} \quad x < 1$

   d) $\displaystyle \frac{5}{16}\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{4^{2n}} \quad |x| < 4$

   e) $\displaystyle \frac{5}{16}\sum_{n=0}^{\infty} \frac{x^{2n}}{4^{2n}} \quad x < 2$

   f) $\displaystyle \frac{5}{16}\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{4^{2n}} \quad |x| < 2$

   g) None of the above.

Solution: d)

5. Which one of the following series is divergent?

a) $\sum \frac{n-1}{n!}$ b) $\sum \frac{n!}{n^n}$ c) $\sum \frac{(n+1)^n}{(2n)^n}$

d) $\sum \frac{(-1)^n}{\ln n}$ e) $\sum \frac{n^6}{(n+1)!}$ f) $\sum \frac{n!}{10^n}$

g) $\sum \frac{3-2^n}{5^n}$ h) they are all convergent

6. What is the sum of the series $\sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{n!}$?

a) $\frac{1}{e^2}$ b) e^2 c) $\cos 2$

d) $\ln 3$ e) $\tan^{-1} 2$ f) -1

g) $\frac{1}{3}$ h) None of the above.

7. Suppose $\sum_{n=0}^{\infty} c_n x^n$ is convergent at $x = 4$ and divergent at $x = -4$. Which of the following statements are true.

I. The radius of convergence is 4.
II. The interval of convergence contains the set $(-4, 4]$ but could be larger.
III. The interval of convergence cannot be determined.
IV. The interval of convergence is $[0, 4]$.

a) Only I. b) Only II. c) Only III. d) Only IV.

e) I. and II. f) II. and III. g) I. and IV.

Solution: a)

8. Evaluate the limit: $\lim_{x\to 0} \frac{1 - \frac{1}{2}x^2 - \cos x}{x^4}$

a) $\frac{1}{2}$ b) $-\frac{1}{2}$ c) $\frac{1}{6}$

d) $-\frac{1}{6}$ e) $\frac{1}{24}$ f) $-\frac{1}{24}$

g) 0 h) The limit does not exist

Free response: Give your answer in the space provided. Answers not placed in this space will be ignored.

9. (8 points) Find the interval of convergence for $\sum_{n=0}^{\infty}(-1)^n \frac{x^n}{n2^n}$.

Solution: Applying the ratio test gives

$$\lim_{n\to\infty}\left|\frac{(-1)^{n+1}\frac{x^{n+1}}{(n+1)2^{n+1}}}{(-1)^n\frac{x^n}{n2^n}}\right| = \lim_{n\to\infty}\frac{n}{2(n+1)}|x| = \frac{|x|}{2} < 1.$$

Thus, $|x| < 2$ and our radius of convergence is 2. Notice that the endpoints are ± 2. Plugging -2 into the series gives

$$\sum_{n=0}^{\infty}(-1)^n\frac{(-1)^n 2^n}{n2^n} = \sum_{n=0}^{\infty}\frac{1}{n}$$

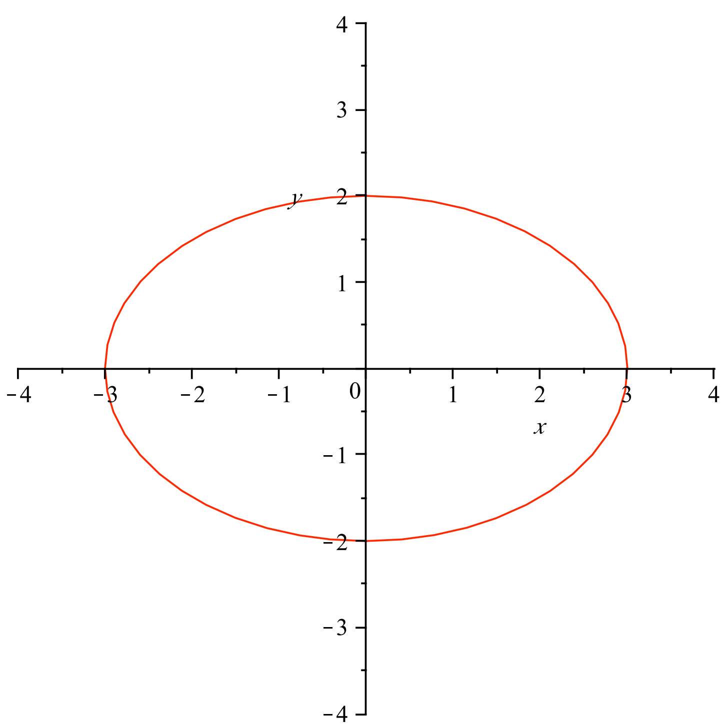
which is the harmonic series, and so diverges. Plugging 2 into the series gives

$$\sum_{n=0}^{\infty}(-1)^n\frac{2^n}{n2^n} = \sum_{n=0}^{\infty}(-1)^n\frac{1}{n}$$

which is the alternating harmonic series and converges. The interval of convergence is therefore $(-2, 2]$.

10. (8 points) A parametric curve is given by $x = 3\sin t$, $y = 2\cos t$ where $-\pi \le t \le \pi$. Sketch the curve by using the parametric equations to plot points. Indicate with an arrow the direction in which the curve is traced as t increases.

Solution:



Direction is clockwise.

11. (8 points) Find a power series representation for the function

$$f(x) = x\tan^{-1}(x)$$

and determine its radius of convergence.

Solution: Since

$$\int \frac{1}{1+x^2}\, dx = \tan^{-1}(x) + C$$

and

$$\frac{1}{1+x^2}\, dx = \sum_{n=0}^{\infty} (-1)^n x^{2n},$$

then

$$\tan^{-1}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}.$$

(The constant is 0 because inverse tangent is 0 at 0). Thus,

$$x \tan^{-1}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+2}.$$

12. (8 points) Find the Maclaurin series for $x^2 \sin 2x$ and give its radius of convergence.

Solution: The Maclaurin series for $\sin(2x)$ is given by

$$\sin(2x) = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1}}{(2n+1)!} x^{2n+1},$$

so

$$x^2 \sin 2x = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1}}{(2n+1)!} x^{2n+3}.$$

13. (8 points) Test the series $\displaystyle\sum_{n=2}^{\infty} \frac{n}{(\ln n)}$ for convergence or divergence.

Solution: You can use the integral test or the comparison test or the divergence test. Notice by L'Hopital's rule that

$$\lim_{x\to\infty} \frac{x}{\ln x} = \lim_{x\to\infty} \frac{1}{\frac{1}{x}} = \lim_{x\to\infty} x = \infty.$$

So, by the divergence test, the series diverges.

14. (10 points) Show that the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ is convergent. How many terms of the series are needed to find the sum to an accuracy of $|error| < 0.01$? Explain your answer.

Solution: Notice that $\sqrt{n} < \sqrt{n+1}$, so $\dfrac{1}{\sqrt{n}} > \dfrac{1}{\sqrt{n+1}}$. Thus, the sequence

$$\{\frac{1}{\sqrt{n}}\}$$

is decreasing and clearly goes to 0. Hence, the series converges by the alternating series test. To estimate the error, we want the next term out to be smaller than 0.01, or

$$\frac{1}{\sqrt{n+1}} < 0.01$$

or

$$\sqrt{n+1} > 100,\ n+1 > 10000,\ n > 9999.$$

15. (8 points) Evaluate the indefinite integral $\int \frac{e^x - 1}{x}\, dx$ as an infinite series.

Solution:

$$\frac{e^x - 1}{x} = \frac{\sum_{n=0}^{\infty} \frac{x^n}{n!} - 1}{x} = \frac{\sum_{n=1}^{\infty} \frac{x^n}{n!}}{x}$$
$$= \sum_{n=1}^{\infty} \frac{x^{n-1}}{n!}.$$

Thus

$$\int \frac{e^x - 1}{x}\, dx = \sum_{n=1}^{\infty} \frac{x^n}{n \cdot n!} + C.$$

16. (10 points) Find the Taylor polynomial of degree 2 for $f(x) = e^{x^2}$ expanded about $a = 0$. Use Taylor's Inequality to estimate the accuracy of the approximation $f(x) \approx T_2(x)$ when x lies in the interval $[0, 0.1]$.

Solution: Note that

$$f'(x) = 2xe^{x^2},$$
$$f''(x) = 2e^{x^2} + 4x^2e^{x^2} = 2e^{x^2}(1 + 2x^2),$$

and

$$f'''(x) = 4xe^{x^2}(1 + 2x^2) + 2e^{x^2}(4x) = 4(2x^3 + 3x)e^{x^2}.$$

Thus, $f(0) = 1$, $f'(0) = 0$, and $f''(0) = 2$. The Taylor polynomial of degree 2 is therefore

$$T_2(x) = 1 + 0 \cdot x + \frac{2}{2!}x^2 = 1 + x^2.$$

(**Note:** You could also have found T_2 by taking the Maclaurin series for e^x, substituting x^2 for x, and taking the terms up to the quadratic.)

Next, we bound the error. To do this, we need a bound of the third derivative (which is why we found it above). Notice that for positive x, each of the terms are positive and increasing, so the third derivative is increasing. A bound for f''' on $[0, 0.1]$ is therefore

$$M = f'''(0.1) = 4(2(.001) + .3)e^{.01} = (1.2008)e^{.01}.$$

Thus, $|R_2| \leq \frac{M(0.1)^3}{3!} = \frac{1.2008e^{0.1}(.001)}{6}$.

END OF EXAM